MATH20132 Calculus of Several Variables. 2020-21

Solutions to Problems 8 Tangent Spaces & Planes

Tangent Planes to level sets.

1. For each of the following level sets find the tangent plane to the surface at the given point $\mathbf{p}$ and give your answer *as a level set.*

i. $(x,y,z)^T \in \mathbb{R}^3 : x^2+y^2+z^2 = 14$ with $\mathbf{p} = (2,1,-3)^T$,

ii. $(x,y,z)^T \in \mathbb{R}^3$:

$$\begin{aligned} x^2+3y^2+2z^2 &= 9, \\ xyz &= -2, \end{aligned}$$

with $\mathbf{p} = (2,-1,1)^T$,

iii. $(x,y,u,v) \in \mathbb{R}^4$:

$$\begin{aligned} x^3-3yu+u^2+2xv &= 12 \\ xv^2+2y^2-3u^2-3yv &= -3. \end{aligned}$$

with $\mathbf{p} = (1,2,-1,2)^T$,

Solution *Theory:* From the notes the Tangent Plane to a point $\mathbf{p}$ on a level set $\mathbf{f}^{-1}(\mathbf{0})$ is

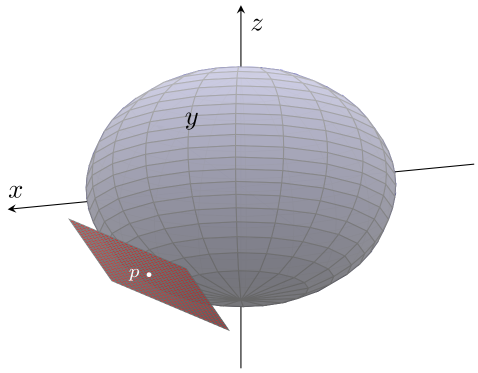
$$\{\mathbf{x} : J\mathbf{f}(\mathbf{p})(\mathbf{x}-\mathbf{p}) = \mathbf{0}\}.$$

i. With $f(\mathbf{x}) = x^2+y^2+z^2-14$ and $\mathbf{p} = (2,1,-3)^T$ the Jacobian matrix is $Jf(\mathbf{p}) = (4,2,-6)$. This is non-zero and so of full-rank and thus the Tangent Plane is those $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$(4,2,-6)\begin{pmatrix} x-2 \\ y-1 \\ z+3 \end{pmatrix} = \mathbf{0},$$

that is $4x+2y-6z-28=0$.

Figure for Part i showing the sphere with the Tangent Plane $z = (4x + 2y - 28)/6$:



ii. The Jacobian of the level set at $\mathbf{p}$ is

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 2x & 6y & 4z \\ yz & xz & xy \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} 4 & -6 & 4 \\ -1 & 2 & -2 \end{pmatrix}.$$

The rows are not linear multiples of each other so $J\mathbf{f}(\mathbf{p})$ is of full rank. Hence the Tangent Plane is those $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{pmatrix} 4 & -6 & 4 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x-2 \\ y+1 \\ z-1 \end{pmatrix} = \mathbf{0},$$

that is

$$\begin{aligned} 2x - 9 - 3y + 2z &= 0, \\ -x + 6 + 2y - 2z &= 0. \end{aligned}$$

iii. The Jacobian of the level set at $\mathbf{p}$ is

$$\begin{pmatrix} 3x^2 + 2v & -3u & -3y + 2u & 2x \\ v^2 & 4y - 3v & -6u & 2xv - 3y \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} 7 & 3 & -8 & 2 \\ 4 & 2 & 6 & -2 \end{pmatrix}.$$

The last two **columns** are linearly independent so $J\mathbf{f}(\mathbf{p})$ is of full rank. Hence the Tangent Plane is those $\mathbf{x} \in \mathbb{R}^4$ satisfying

$$\begin{pmatrix} 7 & 3 & -8 & 2 \\ 4 & 2 & 6 & -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \\ u+1 \\ v-2 \end{pmatrix} = \mathbf{0},$$

which is the level set

$$\begin{aligned} 7x + 3y - 8u + 2v &= 25, \\ 4x + 2y + 6u - 2v &= -2. \end{aligned}$$

2. Return to your answers of Question 1 and write them as graphs instead of level sets. Then give a basis for the Tangent Space.

Solution i. The Tangent Plane was previously given, in Question 1, as those $\mathbf{x} \in \mathbb{R}^3$ satisfying $4x + 2y - 6z - 28 = 0$. This can be written as the graph

$$\left\{ \begin{pmatrix} x \\ y \\ (4x + 2y - 28)/6 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

Since

$$\begin{pmatrix} x \\ y \\ (4x + 2y - 28)/6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -28/6 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ 2/3 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix},$$

a basis for the Tangent Space will be $(1, 0, 2/3)^T$ and $(0, 1, 1/3)^T$ which we could re-scale as $(3, 0, 2)^T$ and $(0, 3, 1)^T$.

ii. The Tangent Plane was previously given as those $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{aligned} 2x - 9 - 3y + 2z &= 0, \\ -x + 6 + 2y - 2z &= 0. \end{aligned}$$

These simultaneous equations can be solved for y and z as functions of x, giving $y = x - 3$ and $z = x/2$. Thus the plane is given by the graph

$$\left\{ \begin{pmatrix} x \\ x - 3 \\ x/2 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

This is the graph of the vector valued function

$$\boldsymbol{\phi}(x) = \begin{pmatrix} x - 3 \\ x/2 \end{pmatrix}.$$

Though called a plane it is geometrically a line.

Writing

$$\begin{pmatrix} x \\ x-3 \\ x/2 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix},$$

we see that $(2,2,1)^T$ is a basis vector for the Tangent Space.

iii The Tangent Plane was previously given as those $\mathbf{x} \in \mathbb{R}^4$ satisfying

$$\begin{aligned} 7x + 3y - 8u + 2v &= 25 \\ 4x + 2y + 6u - 2v &= -2. \end{aligned}$$

Equivalent to solving for u and v is to start with the augmented matrix

$$\left(\begin{array}{cccc|c} 7 & 3 & -8 & 2 & 25 \\ 4 & 2 & 6 & -2 & -2 \end{array} \right).$$

Then apply row operations

$$\underset{r_1 \to r_1 + r_2}{\longrightarrow} \left(\begin{array}{cccc|c} 11 & 5 & -2 & 0 & 23 \\ 4 & 2 & 6 & -2 & -2 \end{array} \right) \underset{r_2 \to r_2 + 3r_1}{\longrightarrow} \left(\begin{array}{cccc|c} 11 & 5 & -2 & 0 & 23 \\ 37 & 17 & 0 & -2 & 67 \end{array} \right)..$$

We could continue to get the identity matrix in the columns corresponding to u and v, but instead we translate the matrix back into equations

$$\begin{aligned} 2u &= 11x + 5y - 23 \\ 2v &= 37x + 17y - 67. \end{aligned}$$

Thus the level set is the graph of the function $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} (11x + 5y - 23)/2 \\ (37x + 17y - 67)/2 \end{pmatrix},$$

where $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$. Looking at the columns in $\left(\mathbf{x}^T, \boldsymbol{\phi}(\mathbf{x})^T\right)^T$ we find that the vectors $(2,0,11,37)^T$ and $(0,2,5,17)^T$ span the Tangent Space.

Tangent Spaces for Image sets.

3. In each case, find parametric equations for the Tangent Plane passing through the point $\mathbf{F}(\mathbf{q})$ on the parametric surfaces given by the following functions.

i. $\mathbf{F}\big((x,y)^T\big) = (x^2+y^2,\, xy,\, 2x-3y)^T, \qquad \text{at } \mathbf{q} = (1,2)^T,$

ii. $\mathbf{F}\big((x,y)^T\big) = (xy^2,\, x^2+y,\, x^3-y^2,\, y^2)^T, \qquad \text{at } \mathbf{q} = (-1,2)^T,$

iii. $\mathbf{F}(t) = (\cos t,\, \sin t,\, t)^T$ at $q = 3\pi$.

Solution *From the Theory:* A result from the notes states that if $J\mathbf{F}(\mathbf{q})$ is of full rank then the tangent plane to the image set of a function is the image set of the Best Affine Approximation to the function.

i. The Jacobian matrix is

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ y & x \\ 2 & -3 \end{pmatrix} \quad \text{so} \quad J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} 2 & 4 \\ 2 & 1 \\ 2 & -3 \end{pmatrix}.$$

The Jacobian matrix $J\mathbf{F}(\mathbf{q})$ is of full rank so the Tangent Plane is the image of the Best Affine Approximation:

$$\begin{aligned} \mathbf{F}(\mathbf{q}) + J\mathbf{F}(\mathbf{q})(\mathbf{x}-\mathbf{q}) &= \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \\ &= \begin{pmatrix} 2x+4y-5 \\ 2x+y-2 \\ 2x-3y \end{pmatrix}, \end{aligned}$$

for $\mathbf{x} \in \mathbb{R}^2$.

Note that the last coordinate function for the Tangent plane is identical to the last one in the definition of $\mathbf{F}$. This should be no surprise since $2x-3y$ is linear.

ii. The Jacobian matrix at $\mathbf{q}$ is

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} y^2 & 2xy \\ 2x & 1 \\ 3x^2 & -2y \\ 0 & 2y \end{pmatrix}_{\mathbf{x}=\mathbf{q}} = \begin{pmatrix} 4 & -4 \\ -2 & 1 \\ 3 & -4 \\ 0 & 4 \end{pmatrix}.$$

It is quickly seen from the last row that the two columns are linearly independent (make sure you understand this) and so $J\mathbf{F}(\mathbf{q})$ is of full-rank. Then the Tangent Plane is the image of the Best Affine Approximation:

$$\begin{aligned}\mathbf{F}(\mathbf{q}) + J\mathbf{F}(\mathbf{q})(\mathbf{x}-\mathbf{q}) &= \begin{pmatrix} -4 \\ 3 \\ -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -2 & 1 \\ 3 & -4 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x+1 \\ y-2 \end{pmatrix} \\ &= \begin{pmatrix} 4x-4y+8 \\ -2x+y-1 \\ 3x-4y+6 \\ 4y-4 \end{pmatrix},\end{aligned}$$

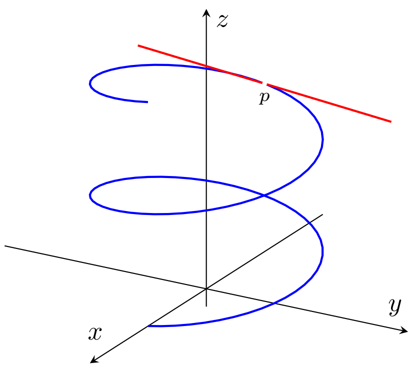
with $\mathbf{x} \in \mathbb{R}^2$.

iii. The Tangent Plane is the image of the Best Affine Approximation:

$$\begin{aligned}\mathbf{F}(q) + J\mathbf{F}(q)(t-q) &= \begin{pmatrix} -1 \\ 0 \\ 3\pi \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} (t-3\pi) \\ &= \begin{pmatrix} -1 \\ -2t+6\pi \\ t \end{pmatrix}.\end{aligned}$$

Though called a tangent **plane** this is geometrically a **line**.

Figure for Question 11iii:



4. Return to Question 7 on Sheet 6. You were asked to show, by using the

Implicit Function Theorem, that the following equations

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0, \end{aligned} \tag{1}$$

determine u and v as functions of x and y for $(x,y)^T$ in an open subset of $\mathbb{R}^2$ containing the point $\mathbf{q} = (-1,1)^T \in \mathbb{R}^2$. The implicit function theorem is an existence result, it does not say what u and v are as functions of x and y. Nonetheless it is possible to find their partial derivatives and you were asked to do this. The answer was

$$\frac{\partial u}{\partial x}(\mathbf{q}) = 0, \ \frac{\partial v}{\partial x}(\mathbf{q}) = 1, \ \frac{\partial u}{\partial y}(\mathbf{q}) = -1 \quad \text{and} \quad \frac{\partial v}{\partial y}(\mathbf{q}) = 0.$$

Use these partial derivatives to find a basis for the tangent space at $\mathbf{p} = (-1,1,1,1)^T$.

Solution The Implicit Function Theorem says that, for $(x,y)^T$ restricted to some set V containing $\mathbf{q}$, the points in the level set lie in the image set of

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x \\ y \\ u(x,y) \\ v(x,y) \end{pmatrix},$$

where $\mathbf{x} = (x,y)^T \in V$. Yet the Tangent space at a point $\mathbf{p} \in S$ on a surface given parametrically as the image of $\mathbf{F}$ is spanned by the columns of $J\mathbf{F}(\mathbf{q})$. In our case the Jacobian at $\mathbf{q}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial u(\mathbf{q})/\partial x & \partial u(\mathbf{q})/\partial y \\ \partial v(\mathbf{q})/\partial x & \partial v(\mathbf{q})/\partial y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the Tangent Space is spanned by $(1,0,0,1)^T$ and $(0,1,-1,0)^T$.

To double check The Jacobian matrix of the system at $\mathbf{p}$ is

$$J\mathbf{f}(\mathbf{p}) = \begin{pmatrix} 2x & 2y & 2v & 2u \\ 3x^2 & 3y^2 & 3u^2 & -3v^2 \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} -2 & 2 & 2 & 2 \\ 3 & 3 & 3 & -3 \end{pmatrix}.$$

The rows of $J\mathbf{f}(\mathbf{p})$ span $T_{\mathbf{p}}(S)^{\perp}$ so we need vectors orthogonal to the rows of $J\mathbf{f}(\mathbf{p})$. It is easily checked that both $(1,0,0,1)^T$ and $(0,1,-1,0)^T$ are orthogonal to all rows of $J\mathbf{f}(\mathbf{p})$. In addition they are linearly independent which means they form a basis for $T_{\mathbf{p}}(S)$.

5. Let $S(\mathbf{u}) = (\cos u \sin v,\ \sin u \sin v,\ \cos v)^T$, where $\mathbf{u} = (u, v)^T$, with $0 \leq v \leq \pi$, $0 \leq u \leq 2\pi$. This is the surface of the unit ball in $\mathbb{R}^3$ in standard spherical coordinates.

i. Show that the tangent space of S at $\mathbf{q} = (\pi,\ \pi/2)^T$ is $T_{\mathbf{p}}S = \mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{p} = S(\mathbf{q})$.

ii. Determine also the tangent space at $\mathbf{q} = (0,\ \pi/4)^T$.

iii. a. Let $\mathbf{w} = (1, 2, -1)^T/\sqrt{6}$. Show that $\mathbf{w} \in T_{\mathbf{p}}S$ where $\mathbf{p} = S\left((0,\ \pi/4)^T\right)$.

b. (Tricky) The definition of $T_{\mathbf{p}}S$ is that $\mathbf{w} \in T_{\mathbf{p}}S$ only if there exists a curve $a : I \to S$ such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. Find a α in this case.

Hint In the notes we prove that $T_{\mathbf{p}}(S) = \{J\mathbf{F}(\mathbf{q})\,\mathbf{x}\}$ when $S = \mathrm{Im}\,\mathbf{F}$. Look at that proof which constructs a curve within the surface.

Solution For a parametrically defined set the tangent space is spanned by the columns of the Jacobian matrix. In this case

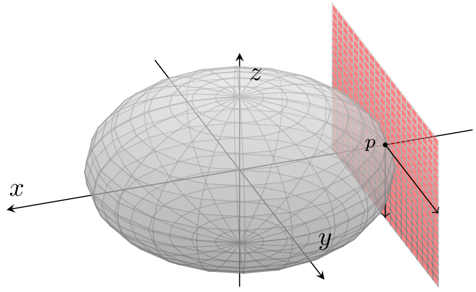
$$JS(\mathbf{u}) = \begin{pmatrix} -\sin u \sin v & \cos u \cos v \\ \cos u \sin v & \sin u \cos v \\ 0 & -\sin v \end{pmatrix}.$$

i. With $\mathbf{q} = (\pi,\ \pi/2)^T$,

$$JS(\mathbf{q}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The columns are $\mathbf{e}_2$ and $-\mathbf{e}_3$ but $\mathrm{Span}(\mathbf{e}_2, -\mathbf{e}_3) = \mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)$ so result follows.

Figure for Part i:

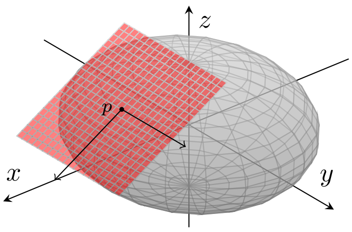


ii. With $\mathbf{q} = (0,\, \pi/4)^T$,

$$JS(\mathbf{q}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

So we can choose $(1, 0, -1)^T$ and $(0, 1, 0)^T$ as a basis for $T_{\mathbf{p}}(S)$.

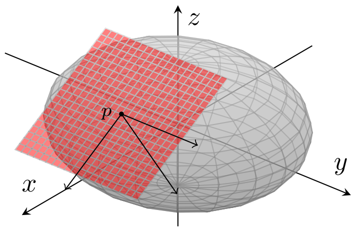
Figure for Part ii:



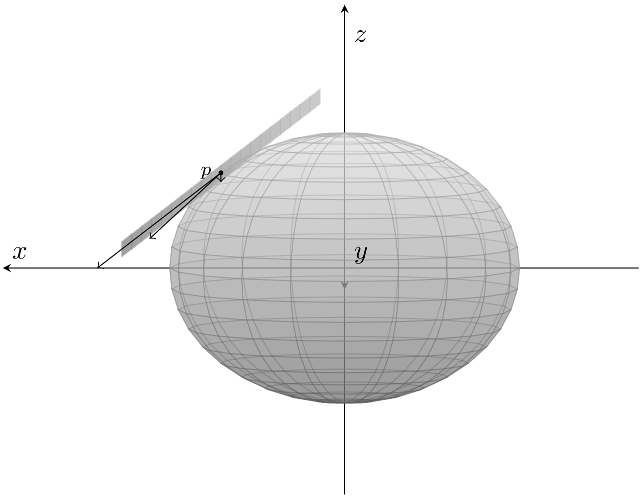
iii. a. The first part follows since

$$\mathbf{w} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{2}{\sqrt{6}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in T_{\mathbf{p}}(S),$$

It might not look obvious from the following figure that $\mathbf{w}$ lies in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$:



But it we change our viewpoint to sideways on to the plane it is more believable.



b. With $\mathbf{v} \in \mathbb{R}^2$ to be chosen our curve will be

$$\alpha : \mathbb{R} \to \mathbb{R}^3,\, t \mapsto S(\mathbf{q} + t\mathbf{v})\,,$$

for $-1 \leq t \leq 1$, say.

By its definition the image of α lies in the surface of the sphere and $\alpha(0) = S(\mathbf{q}) = \mathbf{p}$ as required. The function α is a composition of S and $\mathbf{f}(t) = \mathbf{q} + t\mathbf{v}$ so, to find the tangent vector $\alpha'(t)$, we need apply the Chain Rule.

For a vector-valued function of one variable the derivative equals the Jacobian matrix, so

$$\alpha' v\,(t) = J\alpha(t) = J\,(S \circ \mathbf{f})\,(t) = JS\,(\mathbf{f}(t))\,J\mathbf{f}(t) = JS\,(\mathbf{f}(t))\,\mathbf{f}'\,(t) = JS\,(\mathbf{f}(t))\,\mathbf{v}.$$

Putting $t = 0$ gives

$$\alpha'(0) = JS(\mathbf{f}(0))\,\mathbf{v} = JS(\mathbf{q})\,\mathbf{v}.$$

We require $\alpha'(0) = \mathbf{w}$ so choose $\mathbf{v} = (u, v)^T$ such that $JS(\mathbf{q})\,\mathbf{v} = \mathbf{w}$, i.e.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

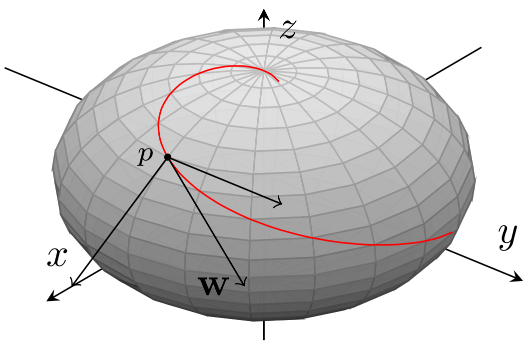
Solve this to find $\mathbf{v} = \left(2/\sqrt{3}, 1/\sqrt{3}\right)^T$. Thus the required curve is

$$t \mapsto S\left(\begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} + \frac{t}{\sqrt{3}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = S\left(\begin{pmatrix} 2t/\sqrt{3} \\ \pi/4 + t/\sqrt{3} \end{pmatrix}\right).$$

That is

$$t \mapsto \begin{pmatrix} \cos\left(2t/\sqrt{3}\right)\sin\left(\pi/4 + t/\sqrt{3}\right) \\ \sin\left(2t/\sqrt{3}\right)\sin\left(\pi/4 + t/\sqrt{3}\right) \\ \cos\left(\pi/4 + t/\sqrt{3}\right) \end{pmatrix}.$$

Figure for Question 14 iiib.



Additional Questions

6 Assume $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a C^1-function on U. Assume that at $\mathbf{a} \in U$ the Jacobian matrix $J\mathbf{f}(\mathbf{a})$ is of full-rank. Prove that there exists an open set $A : \mathbf{a} \in A \subseteq U$ such that $J\mathbf{f}(\mathbf{x})$ is of full-rank for all $\mathbf{x} \in A$.

Solution Assume $n \geq m$. Then $J\mathbf{f}(\mathbf{a})$ has m linearly independent rows. Consider the determinant of the $m \times m$ matrix consisting of these rows in $J\mathbf{f}(\mathbf{x})$. That the rows are linearly independent when $\mathbf{x} = \mathbf{a}$ means the determinant is non-zero when $\mathbf{x} = \mathbf{a}$.

Yet the determinant is a sum of products of the elements of $J\mathbf{f}(\mathbf{x})$, i.e. partial derivatives $\partial f(\mathbf{x})/\partial x^i$. That is, it is a polynomial in these partial derivatives. Yet we are told that f is C^1, i.e. it's partial derivatives are continuous. Hence the polynomial is continuous.

So we have a continuous function, non-zero at $x = \mathbf{a}$. By the properties of continuous functions there exists an open set $A : \mathbf{a} \in A \subseteq U$ such that the polynomial is non-zero in A. That is, the determinant of m rows in $J\mathbf{f}(\mathbf{x})$ is non-zero for all $\mathbf{x} \in A$. In turn this means that $J\mathbf{f}(\mathbf{x})$ has m linearly independent rows, that is it is of full-rank, for all $\mathbf{x} \in A$.

If $m < n$ simply replace row by column in the above argument.

7 Let $C \subseteq \mathbb{R}^3$ be the level set

$$\begin{aligned} x^2z^3 - x^3z^2 &= 0, \\ x^2y + xy^3 &= 2. \end{aligned}$$

Show that in some neighbourhood of $\mathbf{p} = (1,1,1)^T$, C is a curve which can be parametrized by $\mathbf{g}(x) = (x, g_1(x), g_2(x))$ for differentiable functions g_1 and g_2.

Find a parametrization of the Tangent Line to C at $\mathbf{p}$.

Solution The Jacobian matrix of the system at $\mathbf{p}$ is

$$\begin{pmatrix} 2xz^3 - 3x^2z^2 & 0 & 3x^2z^2 - 2x^3z \\ 2xy + y^3 & x^2 + 3xy^2 & 0 \end{pmatrix}_{\mathbf{x}=\mathbf{p}} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 4 & 0 \end{pmatrix}.$$

The last two columns are linearly independent and so the system can be solved in a neighbourhood of $\mathbf{p}$ with the last two variables, y and z, as functions of the first variable, x. In the question $g_1(x) = y(x)$ and $g_2(x) = z(x)$.

The Tangent Line is a special case (only one free parameter) of the Tangent Plane. So the Tangent line is the set of $x \in \mathbb{R}^3$ such that

$$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = \mathbf{0}, \quad \text{that is} \quad \begin{array}{r} -x+z=0 \\ 3x-7+4y=0 \end{array}.$$

Parametrically this is $\mathbf{g}(x) = (x, (7-3x)/4, x)^T, x \in \mathbb{R}$.

Alternative solution If you fail to remember how to find the Tangent Plane for a level set remember instead that the velocity vector at $\mathbf{p} = (1,1,1)^T$ is $\mathbf{v} = (1, g_1'(1), g_2'(1))^T$. Though the Implicit Function Theorem justifies the existence of g_1 and g_2 it does not say what they are. We can, nonetheless, calculate their derivatives. Starting from

$$\begin{aligned} x^2 g_2(x)^3 - x^3 g_2(x)^2 &= 0, \\ x^2 g_1(x) + x g_1(x)^3 &= 2. \end{aligned}$$

Take derivatives

$$\begin{aligned} 2x g_2(x)^3 + 3x^2 g_2(x)^2 g_2'(x) - 3x^2 g_2(x)^2 - 2x^3 g_2(x) g_2'(x) &= 0, \\ 2x g_1(x) + x^2 g_1'(x) + g_1(x)^3 + 3x g_1(x)^2 g_1'(x) &= 0. \end{aligned}$$

Choose $x = 1$, when $g_1(1) = 1$ and $g_2(1) = 1$. Thus

$$\begin{aligned} 2 + 3g_2'(x) - 3 - 2g_2'(x) &= 0, \\ 2 + g_1'(x) + 1 + 3g_1'(x) &= 0. \end{aligned}$$

So $g_1'(1) = -3/4$ and $g_2'(1) = 1$ and the velocity vector is $\mathbf{v} = (1, -3/4, 1)^T$. The Tangent line is $\mathbf{p} + s\mathbf{v}$, i.e.

$$(1+s, 1-3s/4, 1+s)^T$$

where $s \in \mathbb{R}$. The two parametrizations are the same under the mapping $1 + s \leftrightarrow x$.

8 Find the Tangent Plane to the surface

$$\begin{aligned} x^3 - y^3 + xv + uv &= 0, \\ xu^2 + yv^2 &= 0. \end{aligned}$$

where $(x, y, u, v)^T \in \mathbb{R}^4$, at $\mathbf{p} = (-1, 1, -1, -1)^T$. Give your answer as a level set, and also as a graph. Find a basis for the Tangent Space to the surface at $\mathbf{p}$.

Solution The Jacobian of the level set at $\mathbf{p}$ is

$$\left(\begin{array}{cccc} 3x^2 + v & -3y^2 & v & x + u \\ u^2 & v^2 & 2xu & 2yv \end{array} \right)_{\mathbf{x}=\mathbf{p}} = \left(\begin{array}{cccc} 2 & -3 & -1 & -2 \\ 1 & 1 & 2 & -2 \end{array} \right).$$

The last two **columns** are linearly independent so $J\mathbf{f}(\mathbf{p})$ is of full rank. Hence the tangent plane is those $\mathbf{x} \in \mathbb{R}^4$ satisfying

$$\left(\begin{array}{cccc} 2 & -3 & -1 & -2 \\ 1 & 1 & 2 & -2 \end{array} \right) \left(\begin{array}{c} x+1 \\ y-1 \\ u+1 \\ v+1 \end{array} \right) = 0,$$

which is the level set

$$\begin{aligned} 2x - 3y - u - 2v &= -2, \\ x + y + 2u - 2v &= 0. \end{aligned}$$

To find the graph solve for u and v. For example, subtracting the equations gives

$$-x + 4y + 3u = 2.$$

Alternatively, multiply the first equation by 2 and add the two

$$5x - 5y - 6v = -4.$$

Then the Tangent Plane is the graph of the vector-valued function

$$\boldsymbol{\phi}(\mathbf{x}) = \left(\begin{array}{c} (x - 4y + 2)/3 \\ (5x - 5y + 4)/6 \end{array} \right)$$

Then from the columns of

$$\left(\begin{array}{c} \mathbf{x} \\ \boldsymbol{\phi}(\mathbf{x}) \end{array} \right)$$

we find a basis of $(6, 0, 2, 5)^T$ and $(0, 6, -8, -5)^T$ for the Tangent Space.

Check In the notes it was shown that given a level set $\{\mathbf{x} : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$ then the Tangent Space at $\mathbf{p}$ is the set of $\mathbf{x}$ such that $J\mathbf{F}(\mathbf{p})\,\mathbf{x} = \mathbf{0}$. It was noted that the rows of $J\mathbf{F}(\mathbf{p})$ are a basis for $T_{\mathbf{p}}(S)^{\perp}$, the orthogonal complement of $T_{\mathbf{p}}(S)$. So a basis for $T_{\mathbf{p}}(S)$ will all be orthogonal to the rows of $J\mathbf{F}(\mathbf{p})$. Are $(6,0,2,5)^T$ and $(0,6,-8,-5)^T$ orthogonal to the rows of $J\mathbf{f}(\mathbf{p})$? I leave it to the student to check, but it does show that we need never get a question such as this wrong.

9 Find parametric equations for the tangent plane passing through the given point $\mathbf{F}(\mathbf{q})$ on the parametric surfaces given by

i. $\mathbf{F}\big((x,y)^T\big) = (x^2+y^2,\ xy,\ 2x-3y)^T$ at $\mathbf{q} = (1,1)^T$.

ii. $\mathbf{F}\big((s,t)^T\big) = (t\cos s,\ t\sin s,\ t)^T$, $\quad \mathbf{q} = (\pi/2,2)^T$,

iii. $\mathbf{F}\big((s,t)^T\big) = (t^2\cos s,\ t^2,\ t^2\sin s)$, $\quad \mathbf{q} = (0,1)^T$,

Solution i. With $\mathbf{q} = (1,1)^T$, the Best Affine Approximation (and thus the Tangent Plane) is the image set of

$$\begin{aligned}\mathbf{F}(\mathbf{q}) + J\mathbf{F}(\mathbf{q})(\mathbf{x}-\mathbf{q}) &= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ &= \begin{pmatrix} 2x+2y-2 \\ x+y-1 \\ 2x-3y \end{pmatrix},\end{aligned}$$

for $\mathbf{x} \in \mathbb{R}^2$.

ii. the Tangent Plane is $\{\mathbf{x} \in \mathbb{R}^3 : z = y\}$.

iii. the Tangent Plane is $\{\mathbf{x} \in \mathbb{R}^3 : x = y\}$.

10 Find parametric equation for the Tangent Plane passing through the point $\mathbf{F}(\mathbf{q})$ on the parametric surface given by $\mathbf{F}(\mathbf{x}) = (yz, xz, xy, xyz)^T$, for $\mathbf{x} = (x,y,z)^T$ at $\mathbf{q} = (1,-1,2)^T$.

Solution The Jacobian matrix is

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \\ yz & xz & xy \end{pmatrix} \quad \text{so} \quad J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix}.$$

The Jacobian matrix is of full rank so the Tangent Plane is the image of Best Affine Approximation;

$$\begin{aligned}\mathbf{F}(\mathbf{q}) + J\mathbf{F}(\mathbf{q})(\mathbf{x}-\mathbf{q}) &= \begin{pmatrix} -2 \\ 2 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} x-1 \\ y+1 \\ z-2 \end{pmatrix} \\ &= \begin{pmatrix} 2y - z + 2 \\ 2x + z - 2 \\ -x + y + 1 \\ -2x + 2y - z + 4 \end{pmatrix}.\end{aligned}$$

11. Find the tangent planes at the points $\mathbf{p}_1 = (1/\sqrt{2}, 1/4, 1/4)$ and $\mathbf{p}_2 = (\sqrt{3}/2, 0, 1/4)$ on the ellipsoid $x^2 + 4y^2 + 4z^2 = 1$.

Find the line of intersection of these two planes.

Solution Let $f(\mathbf{x}) = x^2 + 4y^2 + 4z^2 - 1$ so the ellipsoid is the level set $f^{-1}(0)$. The Jacobian matrix is $Jf(\mathbf{x}) = (2x, 8y, 8z)$. The Tangent Plane to $f^{-1}(0)$ at $\mathbf{p}_1$ is

$$\begin{aligned} 0 &= Jf(\mathbf{p}_1)(\mathbf{x}-\mathbf{p}_1) = 2\frac{1}{\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right) + 8\frac{1}{4}\left(y - \frac{1}{4}\right) + 8\frac{1}{4}\left(z - \frac{1}{4}\right) \\ &= \sqrt{2}x + 2y + 2z - 2. \end{aligned}$$

That is $\sqrt{2}x + 2y + 2z = 2$.

Similarly the plane at $\mathbf{p}_2$ is $\sqrt{3}x + 2z = 2$.

Solving for y and z and the line of intersection can be given parametrically as

$$\left\{ \left(t, \frac{(\sqrt{3}-\sqrt{2})}{2}t, 1 - t\frac{\sqrt{3}}{2} \right)^T : t \in \mathbb{R} \right\}.$$

12. i. Find the Tangent Plane to the surface $z = xe^y$ at the point $\mathbf{p} = (1,0,1)^T$ on the surface.

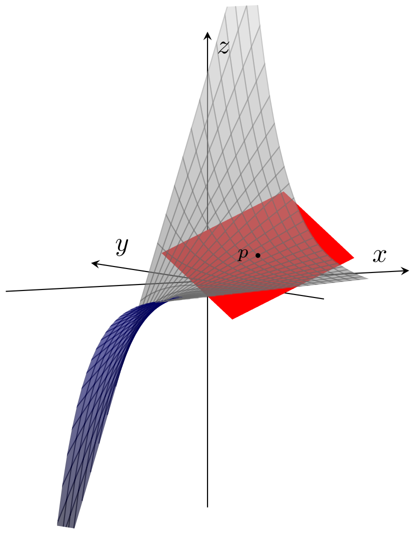
ii. The surfaces $x^2 + y^2 - z^2 = 1$ and $x + y + z = 5$ intersect in a curve Γ. Find the equation in parametric form of the tangent line to Γ at the point $(1,2,2)^T$.

Solution i. Let $f(\mathbf{x}) = xe^y - z$, where $\mathbf{x} = (x, y, z)^T$, so the surface is the level set $f^{-1}(0)$. The Jacobian matrix is $Jf(\mathbf{x}) = (e^y, xe^y, -1)$ so $Jf(\mathbf{p}) = (1, 1, -1)$. The Tangent plane is $Jf(\mathbf{p})(\mathbf{x} - \mathbf{p}) = 0$, that is $\mathbf{x} \in \mathbb{R}^3$ such that

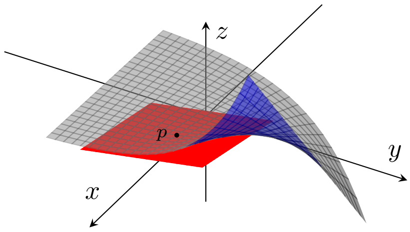
$$0 = (1, 1, -1) \begin{pmatrix} x - 1 \\ y - 0 \\ z - 1 \end{pmatrix} = x + y - z,$$

i.e. $x + y - z = 0$.

Just in case you cannot see the Tangent Plane, here shown under the surface:



I've now changed my viewpoint around by 90° and up a little:



(b) Let $\mathbf{F}(\mathbf{x}) = (x^2 + y^2 - z^2 - 1,\, x + y + z - 5)^T$. Then Γ is the level set $\mathbf{F}^{-1}(\mathbf{0})$. The Jacobian matrix is

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{pmatrix} \quad \text{so} \quad J\mathbf{F}(\mathbf{p}) = \begin{pmatrix} 2 & 4 & -4 \\ 1 & 1 & 1 \end{pmatrix},$$

where $\mathbf{p} = (1, 2, 2)^T$. The Tangent plane to the level set at $\mathbf{p}$ is

$$\left\{\mathbf{x} \in \mathbb{R}^3 : J\mathbf{F}(\mathbf{p})(\mathbf{x} - \mathbf{p}) = 0\right\} = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 2 & 4 & -4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 2 \\ z - 2 \end{pmatrix} = 0\right\}.$$

Thus we get the Tangent plane (here a line) as a level set

$$\begin{aligned} x + 2y - 2z &= 1 \\ x + y + z &= 5. \end{aligned}$$

To give the answer in parametric form solve for y and z. Perhaps $2\times$ second equation add to first so $3x + 4y = 11$, i.e. $y = (11 - 3x)/4$. In the second equation for

$$z = 5 - x - y = (20 - 4x - 11 + 3x)/4 = (9 - x)/4.$$

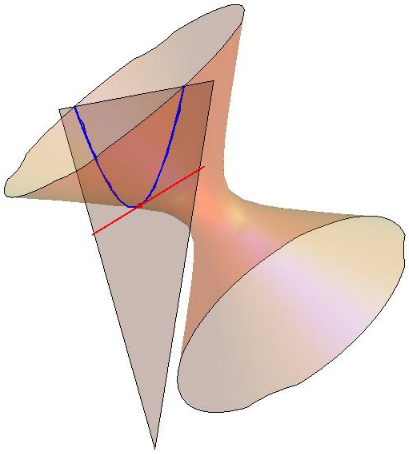
Hence a parametric form for the Tangent line is

$$\left\{\left(x, \frac{11 - 3x}{4}, \frac{9 - x}{4}\right)^T : x \in \mathbb{R}\right\}.$$

You might not like fractions, but a change of variables, $x = 1 + 4t$, gives

$$\left\{(1 + 4t,\, 2 - 3t,\, 2 - t)^T : t \in \mathbb{R}\right\}.$$

Hopefully you can see in the following figure the blue line of intersection of the plane $x + y + z = 5$ with the hyperboloid $x^2 + y^2 - z^2 = 1$, along with the red tangent line:



13. i. Consider the surface $S = \{(x, y, z)^T \in \mathbb{R}^3 : xy = z\}$. Let $\mathbf{p} = (A, B, C)^T$ be a generic point of S. Find the Tangent Plane at $\mathbf{p}$.

ii. Show that the intersection of the Tangent Plane with S consists of two straight lines.

Solution i. If $f(\mathbf{x}) = xy - z$ then $S = f^{-1}(0)$. The Tangent plane for a level set at $\mathbf{p}$ is the set of $\mathbf{x}$ such that $Jf(\mathbf{p})(\mathbf{x} - \mathbf{p}) = 0$. In this case $Jf(\mathbf{p}) = (y, x, -1)_{\mathbf{x}=\mathbf{p}} = (B, A, -1)$. So the plane is the $\mathbf{x} \in \mathbb{R}^3$ such that

$$0 = Jf(\mathbf{p})(\mathbf{x} - \mathbf{p}) = (B,\ A,\ -1)\begin{pmatrix} x - A \\ y - B \\ z - C \end{pmatrix} = Bx + Ay - z - 2AB + C.$$

That is

$$Bx + Ay - z = 2AB - C = AB,$$

since $AB = C$ because $(A, B, C)^T \in S$.

ii. The intersection of the Tangent plane with S consists of $(x, y, z)^T : Bx + Ay - z = AB$ and $xy = z$. Combine as $Bx + Ay - xy = AB$, which rearranges as $(x - A)(y - B) = 0$. Thus we have either $x = A$ or $y = B$.

If $x = A$ then the equation of the surface becomes $Ay = z$ and we get the straight line $\left\{(A, t, At)^T : t \in \mathbb{R}\right\}$.

If $y = B$ we get the straight line $\left\{(t, B, Bt)^T : t \in \mathbb{R}\right\}$.

Does this help as an illustration?